Sojourn time of some reflected Brownian motion in the unit disk

Madalina Deaconu, Mihai Gradinaru, Jean Rodolphe Roche

Sojourn time of some reflected Brownian motion in the unit disk

by

Madalina Deaconu [1], Mihai Gradinaru, Jean Rodolphe Roche
Institut de Mathématiques Elie Cartan, Université Henri Poincaré,
B.P. 239, 54506 Vandœuvre-lès-Nancy Cedex, France

Abstract. - We study the heat diffusion in a domain with an obstacle inside. More precisely, we are interested in the quantity of heat in so far as a function of the position of the heat source at time 0. This quantity is also equal to the expectation of the sojourn time of the Brownian motion, reflected on the boundary of a small disk contained in the unit disk, and killed on the unit circle. We give the explicit expression of this expectation. This allows us to make some numerical estimates and thus to illustrate the behaviour of this expectation as a function of starting point of Brownian motion.

1991 Mathematics Subject Classifications : 60J65, 60H30, 35K20, 35J25, 30C35, 30E25.

Keywords : Reflected Brownian motion, boundary value problems, fractional linear transformation, numerical computations.

Résumé. - Nous étudions la diffusion de la chaleur dans un domaine qui contient un obstacle à l'intérieur. Plus précisement, nous nous intéressons à la quantité de chaleur en tant que fonction de la position de la source de chaleur à l'instant 0. Cette quantité est égale, par ailleurs, à l'espérance du temps de séjour du mouvement brownien dans le disque unité, réfléchi sur la frontière d'un petit disque contenu à l'intérieur du disque unité. Nous donnons une expression explicite de cette espérance. Ceci permet d'avoir des estimations numériques et ainsi d' illustrer le comportement de cette espérance comme fonction de point de départ du mouvement brownien.

Titre en français : *Temps de séjour dans le disque unité d'un certain mouvement brownien réfléchi*

INTRODUCTION

Assume that a heat source is placed in a bounded domain D with a heat absorbing boundary ∂D. Also a small obstacle O with heat reflecting boundary ∂O is placed inside the domain. A natural question arise: given a position of the obstacle, what will be the point $z \in D \setminus O$ where we must place the heat source, such that the quantity of heat

$$Q(z) := \int_{D \setminus O} dw \int_0^\infty dt\, u(t, z, w)$$

will be maximum? It is a classical optimisation problem for linear partial differential equations (see also, [12] and references therein). Here $u(\cdot, z, \cdot)$ is the unique solution of the heat equation

[1] e-mail:Madalina.Deaconu@iecn.u-nancy.fr

with mixed boundary conditions

$$\begin{cases} (\partial u/\partial t)(\cdot,z,\cdot) = (1/2)\Delta u(\cdot,z,\cdot), & \text{in } \mathrm{I\!R}_+^* \times (D \setminus O) \\ u(0,z,\cdot) = \delta_z(\cdot) & \\ (\partial u/\partial \mathrm{n})(\cdot,z,w) = 0, & \text{for } w \in \partial O \\ u(\cdot,z,w) = 0, & \text{for } w \in \partial D. \end{cases}$$

The purpose of this paper is to give an answer to this question, for the simple case where D is the unit disk in the complex plan and O is a small disk inside D, using some probabilistic remarks.

To modelise heat as a probabilistic object, we need to consider the Brownian motion starting from $z \in D$. The reflecting and the absorbing features are ensured by imposing that the stochastic process is reflected on ∂O and killed on ∂D. Then the quantity of heat is related to the sojourn time of this stochastic process in the domain $D \setminus O$. By a simple stochastic calculation we can see that

$$Q(z) = \mathrm{cst.} \cdot \mathrm{E}_z(\tau),$$

where τ is the first hitting time of ∂D by the stochastic process. Hence, finding the quantity of heat is equivalent to finding the preceding expectation (see also [4], Chap. II).

Let us note that we may follow the potential theory point of view: the quantity of heat is the integral on $D \setminus O$ of the fundamental solution (Green-Neumann function or 0-potential) associated to the stochastic process. If $D \setminus O$ is an annulus centered in the origin, the Green function for the Dirichlet problem was explicitely calculated by many authors: [13], p. 140, [3], p. 386, [11], p. 6.41 (see also, [2], §11.7 and notes therein). In [6] (1986) a general Neumann problem is also described (see, § 15.2, 15.7). At our knowledge, theres is no reference for a mixed boundary problem on the annulus.

It must be noticed that, since the expression of $Q(z)$ which we obtain is complicated, in general, computing its maximum is not easy. We illustrate the behaviour of this function using some numerical computations.

1. SETTING AND MAIN RESULT

1.1. Setting

Let us consider a complex Brownian motion B starting from a point z, with $|z| < 1$. It is well known that the expectation of the exit time from the unit disk of B is $(1/2)(1 - |z|^2)$, and it is maximum when the starting point is $z = 0$.

Assume that a reflecting obstacle is placed in the unit disk. What will be the expectation of the exit time from the unit disk of the Brownian motion which is reflected when it hits the obstacle? Let us denote by $(x_t^0 : t \geq 0)$ the process which is a reflected Brownian motion on a inner small circle γ_0, of radius R_0, killed at the first hitting time of the unit circle γ_1.

It is a simple calculation (see also [11], Chap. 6) to show that, if the circles γ_0 and γ_1 are concentric, then the expectation of exit time from the unit disk of x_t^0 is $(1/2)(1-|z|^2+R_0^2 \log |z|^2)$. This expectation is maximum when the starting point lies on the circle γ_0.

We shall also consider the process $(x_t^1 : t \geq 0)$ which is the Brownian motion reflected on the unit circle γ_1 and killed at the first hitting time of γ_0 (see also Figure 1).

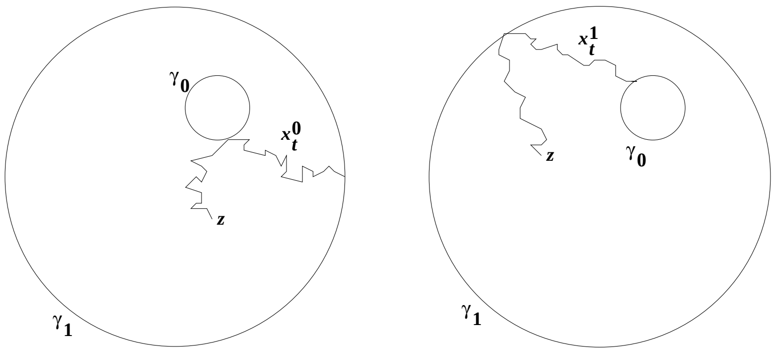

Figure 1: Reflected Brownian motion on γ_0 or on γ_1.

We may assume, without loss of generality, that γ_0 is centered on the real axis. Take $c_0 \in]-1,1[$, $R_0 \in]0, 1-|c_0|[$ and denote the domain between the circles γ_0 and γ_1 by

$$\Omega = \{z \in \mathbb{C} \colon |z| < 1,\ |z - c_0| > R_0\}.$$

For $j = 0$ or 1, we shall denote by τ_j the first hitting time of γ_{1-j} by the process x^j:

$$\tau_j = \inf\{t > 0 : x_t^j \in \gamma_{1-j}\}. \tag{1.1}$$

It is known that, for any $z \in \Omega$, for $j = 0, 1$, τ_j is finite P_z-a.s.

1.2. Main result

We are interested to compute the expectation of the hitting time, $\mathrm{E}_z(\tau_j)$, $j = 0, 1$ as a function of the starting point z. These functions are given in the following main result:

THEOREM 1.1. - *The expectations of the hitting times are given by*

$$\begin{aligned} E_z(\tau_0) &= (1/2)\left(|z\sinh p - \cosh p|^2 - |z\cosh p - \sinh p|^2\right) \\ &- \log\frac{|z\,\cosh p - \sinh p|\,|z\,q\sinh 2p - r|}{|z\,\sinh p - \cosh p|\,|z\,r - q\sinh 2p|} \\ +\frac{R^2}{r^2}\log&\frac{|z\,\cosh p - \sinh p|\,|z\,r - q\sinh 2p|\,|z\,2q(1-q)\sinh 2p - s|}{|z\,\sinh p - \cosh p|\,|z\,q\sinh 2p - r|\,|z\,s - 2q(1-q)\sinh 2p|} + \sum_{n=1}^{\infty} s_n^{(0)}(z,p,R) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} E_z(\tau_1) &= (1/2)\left(|z\sinh p - \cosh p|^2 - |z\cosh p - \sinh p|^2\right) \\ -\log\frac{R\,|z\,\cosh p - \sinh p|}{|z\,\sinh p - \cosh p|\,|z\,r - q\sinh 2p|^2} &+ \frac{R^2}{r^2}\log\frac{|z\,\cosh p - \sinh p|\,|z\,q\sinh 2p - r|}{|z\,\sinh p - \cosh p|\,|z\,s - 2q(1-q)\sinh 2p|} \\ -\frac{q}{r}\frac{|z(1+\cosh 2p) - \sinh 2p|^2 - R^4|z(1-\cosh 2p) + \sinh 2p|^2}{|2z\,r - 2q\sinh 2p|^2} &+ \sum_{n=1}^{\infty} s_n^{(1)}(z,p,R). \end{aligned} \tag{1.3}$$

Here we denoted:

$$p = \frac{1}{4}\log\frac{(1+c_0)^2-R_0^2}{(1-c_0)^2-R_0^2},\ R = \tanh\left(\frac{1}{4}\log\frac{(1+R_0)^2-c_0^2}{(1-R_0)^2-c_0^2}\right), \tag{1.4}$$

$$q = (1-R^2)/2,\ r = \cosh^2 p - R^2\sinh^2 p,\ s = \cosh^2 p - R^4\sinh^2 p. \tag{1.5}$$

The sequences $(s_n^{(j)}(z,p,R))_{n\geq 1}$, *are explicit (see (3.6)) and converge to zero as* R^{4n}, *uniformly for* $z \in \Omega$.

The quantities $s_n^{(j)}$ are sums of logarithms and fractions of the same type as the other terms in (1.2) and (1.3).

Let us note that for $c_0 = 0$, we find the formulas of the concentric circles case. The expectation of the exit time of the Brownian motion from the unit disk (without obstacle) can be obtained taking $c_0 = R_0 = 0$.

1.3. Main ideas and plan

The reflected process that we study can be written, for $j = 0,\ 1$, as

$$x_t^j = B_t - k_t^j, \tag{1.6}$$

with

$$k_0^j = 0,\ k_t^j = \int_0^t \mathrm{n}_j(x_s^j)\, d|k^j|_s,\ |k^j|_t = \int_0^t \mathbb{1}_{(x_s^j\in\gamma_j)} d|k^j|_s, \tag{1.7}$$

where n_j is the outward normal at γ_j to Ω, and, for $j = 0,\ 1$, $(k_t^j : t \geq 0)$ are adapted continuous and locally bounded variation processes (see also, [10], p. 512).

The expectation of the hitting time τ_j is then expressed as a function of the starting point z:

$$E_z(\tau_j) = -2\, H_j(z),\ j = 0,\ 1. \tag{1.8}$$

Indeed, for $j = 0,\ 1$, let H_j be a smooth function which satisfies the mixed boundary conditions (Dirichlet-Neumann) problem

$$\Delta H_j = 1, \ \text{ in } \Omega, \tag{1.9}$$

$$\partial H_j/\partial \mathrm{n}_j = 0, \ \text{ on } \gamma_j, \tag{1.10}$$

$$H_j = 0, \ \text{ on } \gamma_{1-j}. \tag{1.11}$$

By Ito's formula we can write

$$H_j(x_t^j) - H_j(z) = \int_0^t \nabla H_j(x_s^j) dB_s - \int_0^t \nabla H_j(x_s^j)\mathrm{n}_j(x_s^j) d|k^j|_s + (1/2)\int_0^t \Delta H_j(x_s^j) ds.$$

To get (1.8) it suffices to put $t = \tau_j$ and to take the expectation.

Hence, it suffices to compute the function H_j. But, for $j = 0,\, 1$,

$$H_j(z) = \int_\Omega G_j^{(\Omega)}(w,z)\,dw,\ z \in \Omega, \tag{1.12}$$

where $G_j^{(\Omega)}$ is the fundamental solution for the problem (1.9)-(1.11). For $j = 0,\, 1$, $G_j^{(\Omega)}$ satisfies

$$\Delta G_j^{(\Omega)}(w,z) = \delta_w(z),\ \textit{for}\ \ z \in \Omega, \tag{1.13}$$

$$(\partial G_j^{(\Omega)}/\partial \mathrm{n}_j)(w,z) = 0,\ \textit{if}\ \ z \in \gamma_j, \tag{1.14}$$

$$G_j^{(\Omega)}(w,z) = 0,\ \textit{if}\ \ z \in \gamma_{1-j}. \tag{1.15}$$

In (1.13) and (1.14) the differentiation is with respect to z.

If the circles γ_0 and γ_1 are concentric, the expression of this fundamental solution can be obtained as for the reflected linear Brownian motion (see §2). By using a suitable linear fractional transformation we reduce the general case to the case of concentric circles (see §3.2).

The proof of Theorem 1.1 is given in §3.3, except for some technical lemmas postponed to the Appendix. Finally, §4 presents graphs of the expectation as a function of z.

2. CONCENTRIC CIRCLES CASE

2.1. Expectation of the hitting time

In this section we shall assume that $c_0 = 0$, that is γ_0 is centered in the origin. For the sake of completeness, we recall some classical results (see also, [11], Chap. 6).

Proposition 2.1. - *The expectations of the hitting times are given by*

$$\mathrm{E}_z(\tau_0) = (1/2)\,(1 - |z|^2 + R_0^2 \log |z|^2), \tag{2.1}$$

and

$$\mathrm{E}_z(\tau_1) = (1/2)\,(R_0^2 - |z|^2 + \log\,(|z|^2/R_0^2)). \tag{2.2}$$

Remark. We have already noted that (2.1) and (2.2) are nothing but (1.2) and (1.3) with $c_0 = 0$.

Proof of Proposition 2.1. It is classical: for $j = 0, 1$, the function

$$H_j(z) := -(1/4)(a_j - |z|^2 + b_j \log |z|^2),\ a_j, b_j \in \mathbb{R},$$

satisfy (1.9) in the annulus

$$A_{R_0} := \{w \in \mathbb{C} \colon\ R_0 < |w| < 1\}.$$

Then we find a_j, b_j such that the boundary conditions (1.10), (1.11) are fulfiled. Therefore (2.1) and (2.2) are obtained by (1.8). □

REMARK 2.2. - The maximum of the function $z \mapsto \mathrm{E}_z(\tau_0)$ lies on the circle γ_0 and the one of the function $z \mapsto \mathrm{E}_z(\tau_1)$ lies on γ_1. $\square$

Remark. It must be noticed that, in this case we have the "intuition" of the solution for the problem (1.9)-(1.11) because the domain, the annulus A_{R_0}, is very particular.

2.2. Fundamental solution for the annulus

We shall point out the fundamental solution of the problem (1.9)-(1.11) for the annulus A_{R_0}. This function will be denoted, for $j = 0,\, 1$, by $G_j^{(R_0)}$, and it satisfies (1.13)-(1.15) with A_{R_0} instead of Ω.

The idea comes from the study of the linear Brownian motion on $]0,1[$, reflected at 0 and absorbed at 1 (see also, [4], pp 77, 79, or [9], p. 97). This process has the density

$$\sum_{n=-\infty}^{\infty} [q_t(y, -x+4n+2) + q_t(y, x+4n+2) - q_t(y, -x+4n+4) - q_t(y, x+4n)],$$

where $q_t(y,x) := (2\pi t)^{-1/2} \exp(-|x-y|^2/2t)$.

For the 2-dimensional case, we shall use the homothetic transformation $\mu(z) := z/R_0$ and the inversion $\nu(z) := R_0^2/\bar{z}$ instead of the translation and of the symmetry (see also, [11], §4.3). The α-potential associated to the process x^j is obtained by integrating in t the product by $e^{-\alpha t}$ of the following density

$$\sum_{n=-\infty}^{\infty} [p_t(w, (\mu^{4n+2} \circ \nu)(z)) + p_t(w, \mu^{4n+2}(z)) - p_t(w, (\mu^{4n+4} \circ \nu)(z)) - p_t(w, \mu^{4n}(z))],$$

where $p_t(w,z) = (2\pi t)^{-1} \exp(-|w-z|^2/2t)$. It is known that

$$\int_0^\infty e^{-\alpha t} p_t(w,z)dt = (1/2\pi) K_0(\sqrt{2\alpha}\,|w-z|),$$

where K_0 denotes the modified Bessel function of index 0. Moreover, $K_0(\beta) \sim \log(1/\beta)$, as $\beta \downarrow 0$ (see, for instance, [8], p. 133).

Letting $\alpha \downarrow 0$ in the expression of the α-potential for the process x^j, we get the 0-potential or the Green function

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \log \frac{|w - \mu^{4n}(z)|\,|w - (\mu^{4n+4} \circ \nu)(z)|}{|w - (\mu^{4n+2} \circ \nu)(z)|\,|w - \mu^{4n+2}(z)|}.$$

To obtain the solution of (1.13)-(1.15) on A_{R_0}, we need to add a harmonic function (see also, [13], p. 140, [3], p. 386, [11], p. 6.41 for the Dirichlet problem):

PROPOSITION 2.3. - *For $j = 0, 1$ and for $w \neq z$, we introduce*

$$G_j^{(R_0)}(w,z) := \frac{1}{4\pi} \log \frac{|z|^2}{R_0^{2j}} + \frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \log \frac{|w - z/R_0^{4n}|^2 |w - 1/\bar{z}R_0^{4n+2-2j}|^2}{|w - 1/\bar{z}R_0^{4n-2j}|^2 |w - z/R_0^{4n+2}|^2}. \tag{2.3}$$

Then, $G_j^{(R_0)}$ satisfies (1.13)-(1.15) on the annulus A_{R_0}.

Proof. We prove the result for $j = 0$, the case $j = 1$ being similar. The function $G_0^{(R_0)}$ is well defined. Indeed, the series in (2.3) is convergent. For instance, if $n \in \mathbb{N}$, the series behaves as $k \sum_{n=0}^{\infty} R_0^{4n}$, which is convergent since $0 < R_0 < 1$ (here $k = -w/z - w\bar{z}R_0^2 + w\bar{z} + wR_0^2/z$).

Since, for $z \in \gamma_1$, $z\bar{z} = 1$, the general term of the series in (2.3) equals 0 and (1.15) is obvious.

For the proof of (1.14) let us denote $z = \rho e^{i\theta}$. To compute the normal derivative it suffices to differentiate with respect to ρ, since the circles are centered at the origin:

$$(*) \qquad (\partial G_0^{(R_0)}/\partial\rho)(w, \rho\, e^{i\theta}) = 1/(2\pi\rho)$$

$$+\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\frac{-e^{i\theta}/R_0^{4n}}{w-\rho e^{i\theta}/R_0^{4n}}+\frac{e^{i\theta}/\rho^2 R_0^{4n+2}}{w-e^{i\theta}/\rho R_0^{4n+2}}-\frac{e^{i\theta}/\rho^2 R_0^{4n}}{w-e^{i\theta}/\rho R_0^{4n}}+\frac{e^{i\theta}/R_0^{4n+2}}{w-\rho e^{i\theta}/R_0^{4n+2}}\right).$$

Clearly, the series is uniformly convergent. For instance, if $n \in \mathbb{N}$, the series behaves as $k \sum_{n=0}^{\infty} R_0^{4n}$, where $k = w(1+\rho^2)(1-R_0^2)/(\rho^2 e^{i\theta})$. For $z \in A_{R_0}$, $\rho^{-2}+1 < R_0^{-2}+1$, so the series converges.

Then we can verify (1.14), since, by (*) we get,

$$\frac{\partial G_0^{(R_0)}}{\partial\rho}(w, \rho\, e^{i\theta})|_{\rho=R_0} = \frac{1}{2\pi R_0} + \frac{1}{2\pi}\lim_{N\uparrow\infty}\sum_{n=-N}^{n=N}\left(\frac{-e^{i\theta}/R_0^{4n}}{w-e^{i\theta}/R_0^{4n-1}}+\frac{e^{i\theta}/R_0^{4n+4}}{w-e^{i\theta}/R_0^{4n+3}}\right)$$

$$= \frac{1}{2\pi}\frac{1}{R_0} + \frac{1}{2\pi}\lim_{N\uparrow\infty}\left(\frac{-e^{i\theta}/R_0^{-4N}}{w-e^{i\theta}/R_0^{-4N-1}}+\frac{e^{i\theta}/R_0^{4N+4}}{w-e^{i\theta}/R_0^{4N+3}}\right) = 0.$$

Finally to prove (1.13) we note that, for $n \in \mathbb{Z}$, the points λ_n of the form

$$wR_0^{4n},\ 1/wR_0^{4n+2},\ 1/wR_0^{4n},\ wR_0^{4n+2},$$

lie in the complementary of the annulus A_{R_0}, excepting w. Hence, the functions $\log|z-\lambda_n|^2$, $\log|\bar{z}-\lambda_n|^2$ are harmonic in A_{R_0}, and

$$\Delta_z G_0^{(R_0)}(w,z) = (1/2\pi)\Delta_z\left(\log|z-w| + \log|z|\right) = \delta_w(z).$$

$\square$

Remark 2.4. - The function $G_j^{(R_0)}$ is not symmetric in (w, z). However, we can prove that, for $j = 0, 1$,

$$(2.4) \qquad \Delta_w G_j^{(R_0)}(w,z) = \delta_z(w), \ \text{ for } w \in A_{R_0}.$$

Indeed, we note that, for $n \in \mathbb{Z}$, the points of the form

$$z/R_0^{4n},\ 1/\bar{z}R_0^{4n+2-2j},\ 1/\bar{z}R_0^{4n-2j},\ z/R_0^{4n+2}$$

lie in the complementary of the annulus A_{R_0}, excepting z. Then we can proceed as for the proof of (1.13). $\square$

2.3. Find the expectation using the fundamental solution

By integrating (2.3) on A_{R_0} we find the expectation $\mathrm{E}_z(\tau_j)$, $j = 0$, 1, that is (2.1) and (2.2) up to the multiplication by -2 (see, (1.8) and (1.12)).

Let us remark first, that the integrals of the terms of the series are zero except for the term corresponding to $n = 0$. Indeed, this is a consequence of the facts that, for $j = 0, 1$,

$$|z/R_0^{4n}|,\ |1/\bar{z}R_0^{4n+2-2j}|,\ |1/\bar{z}R_0^{4n-2j}|,\ |z/R_0^{4n+2}| \text{ are } \begin{cases} > 1, & \text{if } n \in \mathbb{Z}_+^* \\ < R_0, & \text{if } n \in \mathbb{Z}_-^*, \end{cases}$$

and of the following:

Lemma 2.5. - *We have*

$$\int_{A_{R_0}} \log|\zeta - \lambda|^2 d\zeta = \begin{cases} 2\pi(1 - R_0^2)\log|\lambda|, & \text{if } |\lambda| > 1 \\ -2\pi R_0^2 \log|\lambda| - \pi(1 - |\lambda|^2), & \text{if } R_0 < |\lambda| < 1 \\ -2\pi R_0^2 \log R_0 - \pi(1 - R_0^2), & \text{if } |\lambda| < R_0. \end{cases} \tag{2.5}$$

The proof of this lemma is postponed in the Appendix.

On the other hand,

$$\int_{A_{R_0}} G_j^{(R_0)}(w,z)dw = \frac{1}{4\pi}\int_{A_{R_0}} \log\frac{|z|^2}{R_0^{2j}}dw + \frac{1}{4\pi}\int_{A_{R_0}} \log\frac{|w-z|^2|w-1/\bar{z}R_0^{2-2j}|^2}{|w-1/\bar{z}R_0^{-2j}|^2|w-z/R_0^2|^2}dw.$$

Since, for $j = 0$, $R_0 < |z| < 1$, $|z|/R_0^2 > 1$, $1/|\bar{z}|R_0^2 > 1$ and $1/|\bar{z}| > 1$, using again (2.5), we get

$$\begin{aligned}\int_{A_{R_0}} G_0^{(R_0)}(w,z)dw &= (1/2\pi)\log|z|\int_{A_{R_0}} dw + (1/4\pi)\Big(-2\pi R_0^2\log|z| - \pi(1-|z|^2) \\ &\quad + 2\pi(1-R_0^2)\log(1/|\bar{z}|R_0^2) - 2\pi(1-R_0^2)\log(1/|\bar{z}|) - 2\pi(1-R_0^2)\log(|\bar{z}|/R_0^2)\Big) \\ &= (1/2)(1-R_0^2)\log|z| + (1/4)\Big(|z|^2 - 1 - 2\log|z|\Big) = (-1/2)\,\mathrm{E}_z(\tau_0).\end{aligned}$$

The calculation is similar for $j = 1$, by noting that $R_0^2/|\bar{z}| < R_0$. □

3. FRACTIONAL LINEAR TRANSFORMATION AND GENERAL CASE

3.1. Fractional linear transformation

Let us consider, for $t \in \mathbb{R}$, the transformation:

$$a_t(\zeta) := \frac{\zeta\cosh t + \sinh t}{\zeta\sinh t + \cosh t},\ \zeta \in \mathbb{C}. \tag{3.1}$$

This is an one-parameter transformation group: $a_t a_s = a_{t+s}$. We recall below some well known properties of these transformations.

Remark 3.1. - The image of the circle centered in the origin, with radius $\tanh r$ by the fractional linear transformation a_t is a circle centered on the real axis. Moreover, this transformation leaves the unit circle invariant.

Indeed, it is no difficult to see that $|a_t(\zeta)| = 1$, provided $|\zeta| = 1$. On the other hand, by classical properties of the fractional linear transformations, the image of an orthogonal circle to the real axis is a circle orthogonal to the real axis. Therefore, the center of the image circle lies on the real axis.

Moreover, the images of the points $\tanh r$ and $-\tanh r$ are, respectively, $\tanh(t+r)$ and $\tanh(t-r)$. Hence, the image circle has the center $(1/2)(\tanh(t+r) + \tanh(t-r))$ and the radius, $(1/2)(\tanh(t+r) - \tanh(t-r))$.

In particular, the circle γ_0, centered in c_0, with radius R_0, is the image of a circle centered in 0 having the radius R, by the transformation of parameter p, where R, p are given by (1.4).□

REMARK 3.2. - Let α and α^{-1} be the solutions of the equation

$$\alpha + \alpha^{-1} = (1 + c_0^2 - R_0^2)/c_0.$$

By a geometric reasoning we can see that, for $j = 0, 1$,

$$\left|(\zeta - \alpha)/(\zeta - \alpha^{-1})\right| = \text{cst. , provided } \zeta \in \gamma_j. \tag{3.2}$$

Therefore, γ_0 and γ_1 are in the family of Apollonius circles with respect to α and α^{-1} (see, [1], p. 84). There exists an orthogonal family of circles which pass through the points α and α^{-1}, and satisfies

$$\arg(\zeta - \alpha)/(\zeta - \alpha^{-1}) = \text{cst.} \tag{3.2'}$$

(see also, [7], p. 53). □

3.2. Fundamental solution for the general domain

We shall point out the fundamental solution for the problem (1.9)-(1.11) using the one on the annulus A_{R_0} and the linear transformation defined above(see also, [11], p. 6.19):

PROPOSITION 3.3. - *For $j = 0, 1$ and for $w \neq z$ we introduce*

$$G_j^{(\Omega)}(w, z) := G_j^{(R)}(a_{-p}(w), a_{-p}(z)), \tag{3.3}$$

where $G_j^{(R)}$ is given by (2.3) *with the parameter R given in* (1.4). *Then $G_j^{(\Omega)}$ satisfies* (1.13)-(1.15).

Proof. Let us first notice that

$$a_{-p}(z) \in A_R = \{\zeta \in \mathbb{C} : R < |\zeta| < 1\}, \text{ provided } z \in \Omega \tag{*}$$

(see also, [11], p. 4.29). It follows, by a similar reasoning as for the proof of Proposition 2.3, that

$$\Delta_z G_j^{(\Omega)}(w, z) = \Delta_z G_j^{(R)}(a_{-p}(w), a_{-p}(z))$$
$$= (1/2\pi)\Delta_z \left(\log|a_{-p}(z) - a_{-p}(w)| + \log(|a_{-p}(z)|/R^j)\right).$$

Since a_t is a holomorphic function, the second term in the above sum is a harmonic function. Then, by (3.1), we get

$$|a_{-p}(z) - a_{-p}(w)| = |z - w|/(|-z\sinh p + \cosh p|\,|-w\sinh p + \cosh p|).$$

Therefore,

$$\Delta_z G_j(w,z) = (1/2\pi)\Delta_z\left(\log|z-w| - \log|(1/\tanh p) - z|\right)$$

and we get (1.13), since $\tanh p < 1$.

In order, to obtain (1.14) we use the corresponding property of $G_j^{(R)}$, (*) and Remark 3.2. Indeed, to calculate the derivative of $G_j^{(\Omega)}$ in the normal direction to the circle γ_j we can use an infinitesimal shift along an arc of circle from the orthogonal family given by (3.2'). We have already observed that a_{-p} send the Apollonius circles (3.2) in the concentric circles in 0 and the orthogonal family (3.2') in straight lines through 0 (that is, into another two orthogonal families of circles in wider sense, see, [1], p. 79). Therefore, using an infinitesimal shift along a segment on a line through 0, the upper derivative is equal to the derivative of $G_j^{(R)}$ in the normal direction to the image circle centered in 0. But this derivative is zero and (1.14) follows.

Finally, (1.15) is a consequence of (*) and the corresponding property for $G_j^{(R)}$. □

3.3. Expectation of the hitting time: proof of Theorem 1.1

As in §2.3 we shall find the expectation of the hitting time using (1.8), (1.12) and the fundamental solution obtained in Proposition 3.3.

By (3.3) and (*) from the proof of Proposition 3.3, we can write, for $j = 0$, 1,

$$H_j(z) = \int_\Omega G_j^{(R)}(a_{-p}(w), a_{-p}(z))dw = \int_{A_R} G_j^{(R)}(\zeta, a_{-p}(z))|\mathrm{Jac}(\zeta)|d\zeta,$$

where $\mathrm{Jac}(\zeta)$ is the Jacobian of the transformation a_p. Let us write this Jacobian as

$$\mathrm{Jac}(\zeta) = \frac{1}{|\zeta\sinh p + \cosh p|^4} = \frac{1}{\sinh^4 p}\frac{1}{|\zeta - k|^4} = \frac{1}{4\sinh^4 p}\cdot\Delta_\zeta\left(\frac{1}{|\zeta-k|^2}\right),$$

where $k = -1/\tanh p < -1$. Therefore,

$$H_j(z) = \frac{1}{4\sinh^4 p}\int_{A_R} G_j^{(R)}(\zeta, a_{-p}(z))\Delta_\zeta\left(\frac{1}{|\zeta-k|^2}\right)d\zeta.$$

Using Green's formula, we obtain

(i) $$H_j(z) = \frac{1}{4\sinh^4 p}\int_{A_R}\Delta_\zeta G_j^{(R)}(\zeta, a_{-p}(z))\cdot\frac{1}{|\zeta-k|^2}d\zeta$$

$$+\frac{1}{4\sinh^4 p}\int_{\partial A_R}\left(G_j^{(R)}(\zeta, a_{-p}(z))\frac{\partial}{\partial \mathrm{n}}\frac{1}{|\zeta-k|^2} - \frac{1}{|\zeta-k|^2}\frac{\partial}{\partial \mathrm{n}}G_j^{(R)}(\zeta, a_{-p}(z))\right)d\sigma := \Lambda_1^{(j)} + \Lambda_2^{(j)},$$

where the normal derivatives and the surface integral are with respect to ζ.

By (2.4), the first integral in (i) is the same for $j = 0$, 1, and equals to

(ii) $$\Lambda_1 = \frac{1}{4\sinh^4 p} \cdot \frac{1}{|a_{-p}(z) - k|^2} = \frac{1}{4\sinh^4 p} \cdot \frac{\tanh^2 p}{|1 + a_{-p}(z)\tanh p|^2}.$$

By (2.3), the second integral in (i) can be written as

(iii) $$\Lambda_2^{(j)} = \frac{1}{16\pi\sinh^4 p}\int_{\partial A_R}\left(\log\frac{|a_{-p}(z)|^2}{R^{2j}}\frac{\partial}{\partial \mathrm{n}}\frac{1}{|\zeta-k|^2} - \frac{1}{|\zeta-k|^2}\frac{\partial}{\partial \mathrm{n}}\log\frac{|a_{-p}(z)|^2}{R^{2j}}\right)d\sigma$$
$$+\frac{1}{16\pi\sinh^4 p}\sum_{n=-\infty}^{\infty}\int_{\partial A_R}\left(t_{j,n}(\zeta, a_{-p}(z), R)\frac{\partial}{\partial \mathrm{n}}\frac{1}{|\zeta-k|^2} - \frac{1}{|\zeta-k|^2}\frac{\partial}{\partial \mathrm{n}}t_{j,n}(\zeta, a_{-p}(z), R)\right)d\sigma$$
$$:= \Lambda_{21}^{(j)} + \Lambda_{22}^{(j)},$$

where

$$t_{j,n}(\zeta, a_{-p}(z), R) = \log\frac{|\zeta - a_{-p}(z)/R^{4n}|^2|\zeta - 1/\bar{a}_{-p}(z)R^{4n+2-2j}|^2}{|\zeta - 1/\bar{a}_{-p}(z)R^{4n-2j}|^2|\zeta - a_{-p}(z)/R^{4n+2}|^2}.$$

Since the derivatives are in ζ, the first integral in (iii) equals to

$$\Lambda_{21}^{(j)} = \frac{1}{16\pi\sinh^4 p}\left(\int_{\{|\zeta|=1\}}\frac{\partial}{\partial \mathrm{n}}\frac{1}{|\zeta-k|^2}\,d\sigma - \int_{\{|\zeta|=R\}}\frac{\partial}{\partial \mathrm{n}}\frac{1}{|\zeta-k|^2}\,d\sigma\right)\log\frac{|a_{-p}(z)|^2}{R^{2j}}.$$

To compute the preceding quantity we shall use the following:

Lemma 3.4. - *For* $0 < \rho \le 1$,

(3.4) $$\frac{1}{4\pi}\int_{\{|\zeta|=\rho\}}\frac{\partial}{\partial \mathrm{n}}\frac{1}{|\zeta-k|^2}\,d\sigma = \frac{\rho^2}{(k^2-\rho^2)^2}.$$

We postpone the proof of this lemma in the Appendix, and we use its result to obtain

(iv) $$\Lambda_{21}^{(j)} = \frac{1}{4}\left(1 - \frac{R^2}{(k^2-R^2)^2/(k^2-1)^2}\right)\log\frac{|a_{-p}(z)|^2}{R^{2j}} = \frac{1}{4}\left(1 - \frac{R^2}{r^2}\right)\log\frac{|a_{-p}(z)|^2}{R^{2j}},$$

since $k^2 - 1 = 1/\sinh^2 p$ and here r is given by (1.5).

The second term in (iii) can be written as

(v) $$\Lambda_{22}^{(j)} = \frac{1}{16\pi\sinh^4 p}\sum_{n=-\infty}^{\infty}\Big\{\Big(J(1, a_{-p}(z)/R^{4n}) - J(R, a_{-p}(z)/R^{4n})\Big)$$
$$+\Big(J(1, 1/\bar{a}_{-p}(z)R^{4n+2-2j}) - J(R, 1/\bar{a}_{-p}(z)R^{4n+2-2j})\Big)$$
$$-\Big(J(1, 1/\bar{a}_{-p}(z)R^{4n-2j}) - J(R, 1/\bar{a}_{-p}(z)R^{4n-2j})\Big)$$
$$-\Big(J(1, a_{-p}(z)R^{4n+2}) - J(R, a_{-p}(z)R^{4n+2})\Big)\Big\} := \sum_{n=-\infty}^{\infty}\Lambda_{22n}^{(j)},$$

where

(vi) $$J(\rho, \lambda) := \int_{\{|\zeta|=\rho\}}\left(\log|\zeta-\lambda|^2\frac{\partial}{\partial \mathrm{n}}\frac{1}{|\zeta-k|^2} - \frac{1}{|\zeta-k|^2}\frac{\partial}{\partial \mathrm{n}}\log|\zeta-\lambda|^2\right)d\sigma.$$

The expression of $J(\rho, \lambda)$ is contained in the following result, which proof will be given in the Appendix:

LEMMA 3.5 - *For* $0 < \rho \leq 1$,

$$J(\rho, \lambda) = \begin{cases} \frac{8\pi\rho^2}{(k^2-\rho^2)^2} \log|\lambda - \rho^2/k|, & \text{if } |\lambda| > \rho \\ \frac{8\pi\rho^2}{(k^2-\rho^2)^2} \log \rho|1 - \lambda/k| - \frac{4\pi}{k^2-\rho^2} \frac{1-|\lambda/k|^2}{|1-\lambda/k|^2}, & \text{if } |\lambda| < \rho. \end{cases} \tag{3.5}$$

Recalling that $1/k = -\tanh p$, by (vi) and (3.5) we can compute the general term of the series in (v)

$$\frac{J(1,\lambda) - J(R,\lambda)}{16\pi \sinh^4 p} = \begin{cases} \frac{1}{2}\log|\lambda + \tanh p| - \frac{R^2}{2r^2}\log|\lambda + R^2 \tanh p|, & \text{if } |\lambda| > 1, \\ \frac{1}{2}\log|1 + \lambda \tanh p| - \frac{R^2}{2r^2}\log|\lambda + R^2\tanh p| & \\ \qquad - \frac{1}{4\sinh^2 p} \cdot \frac{1-|\lambda|^2\tanh^2 p}{|1+\lambda \tanh p|^2}, & \text{if } R < |\lambda| < 1 \\ \frac{1}{2}\log|1 + \lambda\tanh p| - \frac{R^2}{2r^2}\log R|1 + \lambda \tanh p| & \\ \qquad - \frac{q}{2r} \cdot \frac{1-|\lambda|^2 \tanh^2 p}{|1+\lambda\tanh p|^2}, & \text{if } |\lambda| < R. \end{cases} \tag{vii}$$

Here q, r are given by (1.5) and

$$\lambda \in \left\{a_{-p}(z)/R^{4n}, 1/\bar{a}_{-p}(z)R^{4n+2-2j}, 1/\bar{a}_{-p}(z)R^{4n-2j}, a_{-p}(z)/R^{4n+2} : n \in \mathbb{Z}, j = 0,\, 1\right\}.$$

In order, to apply (vii) for the calculation of $\Lambda^{(j)}_{22n}$, we need to study the modulus of the complex in the preceding set, for $n \in \mathbb{Z}$ and for $j = 0$, 1.

For $n = 0$,

$$R < |a_{-p}(z)| < 1, |1/\bar{a}_{-p}(z)R^2| > 1, |1/\bar{a}_{-p}(z)| > 1, |1/\bar{a}_{-p}(z)R^{-2}| < R, |a_{-p}(z)/R^2| > 1.$$

Therefore,

$$\Lambda^{(0)}_{220} = \frac{1}{2}\log\frac{|1 + a_{-p}(z)R^2 \tanh p|}{|a_{-p}(z) + R^2 \tanh p|} - \frac{1}{4\sinh^2 p} \cdot \frac{1 - |a_{-p}(z)|^2 \tanh^2 p}{|1 + a_{-p}(z)\tanh p|^2} \tag{viii}$$

$$-\frac{R^2}{2r^2}\log\frac{|a_{-p}(z) + R^2 \tanh p|\,|1 + a_{-p}(z)R^4 \tanh p|}{|1 + a_{-p}(z)R^2 \tanh p|\,|a_{-p}(z) + R^4 \tanh p|}$$

and

$$\Lambda^{(1)}_{220} = \frac{1}{2}\log\frac{R^2|1 + a_{-p}(z)\tanh p|^2}{|a_{-p}(z) + R^2\tanh p|^2} - \frac{1}{4\sinh^2 p} \cdot \frac{1 - |a_{-p}(z)|^2\tanh^2 p}{|1 + a_{-p}(z)\tanh p|^2} \tag{ix}$$

$$-\frac{R^2}{2r^2}\log\frac{R|1+a_{-p}(z)R^2\tanh p|}{|a_{-p}(z)+R^4\tanh p|}+\frac{q}{2r}\cdot\frac{|a_{-p}(z)|^2-R^4\tanh^2 p}{|a_{-p}(z)+R^2\tanh p|^2}.$$

For $n\in\mathbb{Z}_+^*$ and $j=0,\,1$,

$$|a_{-p}(z)/R^{4n}|,|1/\bar{a}_{-p}(z)R^{4n+2-2j}|,|1/\bar{a}_{-p}(z)R^{4n-2j}|,|a_{-p}(z)/R^{4n+2}|>1,$$

while, for $n\in\mathbb{Z}_-^*$ and $j=0,\,1$,

$$|a_{-p}(z)/R^{4n}|,|1/\bar{a}_{-p}(z)R^{4n+2-2j}|,|1/\bar{a}_{-p}(z)R^{4n-2j}|,|a_{-p}(z)/R^{4n+2}|<R.$$

Thus, for $n\in\mathbb{Z}_+^*$ and $j=0,\,1$, the computations gives:

$$\text{(xi)}\qquad \Lambda_{22n}^{(j)}+\Lambda_{22(-n)}^{(j)}=\frac{1}{2}\Bigg(\log\frac{|a_{-p}(z)+R^{4n}\tanh p|\,|1+a_{-p}(z)\,R^{4n+2-2j}\tanh p|}{|1+a_{-p}(z)\,R^{4n-2j}\tanh p|\,|a_{-p}(z)+R^{4n+2}\tanh p|}$$

$$+\log\frac{|1+a_{-p}(z)\,R^{4n}\tanh p|\,|a_{-p}(z)+R^{4n-2+2j}\tanh p|}{|a_{-p}(z)+R^{4n+2j}\tanh p|\,|1+a_{-p}(z)\,R^{4n-2}\tanh p|}\Bigg)$$

$$-\frac{R^2}{2r^2}\Bigg(\log\frac{|a_{-p}(z)+R^{4n+2}\tanh p|\,|1+a_{-p}(z)\,R^{4n+4-2j}\tanh p|}{|1+a_{-p}(z)\,R^{4n+2-2j}\tanh p|\,|a_{-p}(z)+R^{4n+4}\tanh p|}$$

$$+\log\frac{|1+a_{-p}(z)\,R^{4n}\tanh p|\,|a_{-p}(z)+R^{4n-2+2j}\tanh p|}{|a_{-p}(z)+R^{4n+2j}\tanh p|\,|1+a_{-p}(z)\,R^{4n-2}\tanh p|}\Bigg)$$

$$-\frac{q}{2r}\Bigg(\frac{1-|a_{-p}(z)|^2R^{8n}\tanh^2 p}{|1+a_{-p}(z)\,R^{4n}\tanh p|^2}+\frac{|a_{-p}(z)|^2-R^{8n-4+4j}\tanh^2 p}{|a_{-p}(z)+R^{4n-2+2j}\tanh p|^2}$$

$$-\frac{|a_{-p}(z)|^2-R^{8n+4j}\tanh^2 p}{|a_{-p}(z)+R^{4n+2j}\tanh p|^2}-\frac{1-|a_{-p}(z)|^2R^{8n-4}\tanh^2 p}{|1+a_{-p}(z)\,R^{4n-2}\tanh p|^2}\Bigg),$$

where q, r are given by (1.5) and $a_{-p}(z)=(z\cosh p-\sinh p)/(-z\sinh p+\cosh p)$.

Combining (i), (ii), (iv), (vii) and (xi) we get

$$\text{(xii)}\qquad H_0(z)=\Lambda_1+\Lambda_{21}^{(0)}+\Lambda_{220}^{(0)}+\sum_{n=1}^{\infty}(\Lambda_{22n}^{(0)}+\Lambda_{22(-n)}^{(0)}),$$

while, from (i), (ii), (iv), (ix) and (3.6) we get

$$\text{(xiii)}\qquad H_1(z)=\Lambda_1+\Lambda_{21}^{(1)}+\Lambda_{220}^{(1)}+\sum_{n=1}^{\infty}(\Lambda_{22n}^{(1)}+\Lambda_{22(-n)}^{(1)}).$$

Thus, (1.2)-(1.3) are obtained using (1.8) and (xi), where we have denoted by

$$\text{(3.6)}\qquad s_n^{(j)}(z,p,R):=-2\,(\Lambda_{22n}^{(j)}+\Lambda_{22(-n)}^{(j)}),\ n\in\mathbb{N}^*,\ j=0,\,1.$$

In order, to end the proof, let us show that for $j = 0$, 1, the series with general term given by (3.6) are convergent. For instance,

$$\sum_{n=1}^{\infty} \log \frac{|a_{-p}(z) + R^{4n} \tanh p|\,|1 + a_{-p}(z)\, R^{4n+2-2j} \tanh p|}{|1 + a_{-p}(z)\, R^{4n-2j} \tanh p|\,|a_{-p}(z) + R^{4n+2} \tanh p|}$$

has the same behaviour as $k \sum_{n=1}^{\infty} R^{4n}$, where $k = (1/a_{-p}(z) + a_{-p}(z)R^{2-2j} - a_{-p}(z)R^{-2j} - R^2/a_{-p}(z)) \cdot \tanh p$. The other series with logarithm can be treated in the same manner. Similarly,

$$\sum_{n=1}^{\infty} \left(\frac{1 - |a_{-p}(z)|^2 R^{8n} \tanh^2 p}{|1 + a_{-p}(z)\, R^{4n} \tanh p|^2} + \frac{|a_{-p}(z)|^2 - R^{8n-4+4j} \tanh^2 p}{|a_{-p}(z) + R^{4n-2+2j} \tanh p|^2} - \frac{|a_{-p}(z)|^2 - R^{8n+4j} \tanh^2 p}{|a_{-p}(z) + R^{4n+2j} \tanh p|^2} - \frac{1 - |a_{-p}(z)|^2 R^{8n-4} \tanh p}{|1 + a_{-p}(z)\, R^{4n-2} \tanh p|^2} \right)$$

behaves as $k \sum_{n=1}^{\infty} R^{4n}$, where $k = (-2 \tanh p)(1 - R^{-2})(|a_{-p}(z)| - R^{2j}/|a_{-p}(z)|)$. This ends the proof of the theorem except for the proofs of the lemmas. □

4. NUMERICAL RESULTS

In this section we present numerical approximations of the analytical representation of the expectation, given in (1.2) or (1.3). We plot two views of the expectation as a function of the starting point: three-dimensional graph and its vertical section.

Figures 2 and 3 illustrate the case $j = 0$. Recall that for the concentric circles case the maximum lies on γ_0. Suppose now that $c_0 \neq 0$. When R_0 is small the position of the maximum is close to zero as we can see in Figure 2. The increase of R_0 gives a displacement of this position towards γ_1. This is quite different with respect to a deterministic motion in which is quite natural to think that z must be far from this circle.

Figures 5 and 6 correspond to the case $j = 1$. The maximum lies on γ_1 and its value is a decreasing function with respect to R_0.

In parallel to semi-analytical method present in this paper we consider a classical finite element method to solve the partial differential equation (1.9), (1.10) and (1.11). A solution of this problem is in $\mathrm{H}^2(\Omega)$, the Sobolev space of function u such that u, ∇u and Δu belongs to $\mathrm{L}^2(\Omega)$. By the finite element we compute a approximation of H_j belonging to a sub-space V_h of continuous function piecewise linear in Ω. The result obtained by the finite element method are similar to those obtained by the semi-analytical method considered in this paper. One difference between the two methods is that the maximum obtained by the finite element method is a approximation with an error of order h^2, h being the discretisation parameter. We represent in figure 4 and figure 7 results computed by the finite element technique, which correspond, respectively to figure 2b and figure 6b.

a. b.

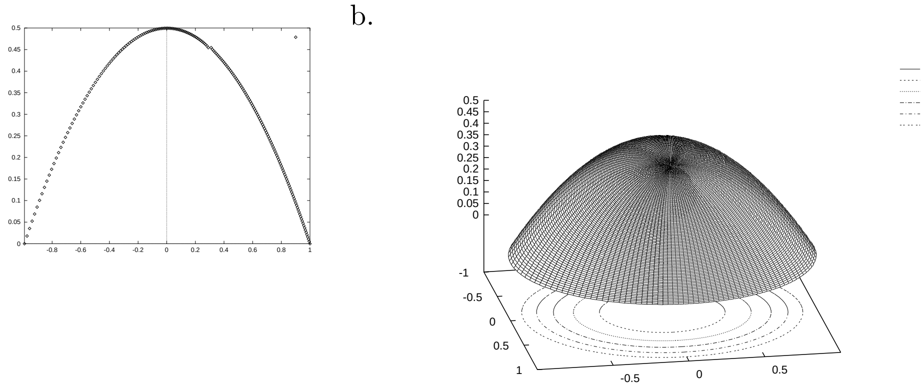

Figure 2: Case $j = 0$ with $c_0 = 0.1$ and $R_0 = 0.01$.

a. b.

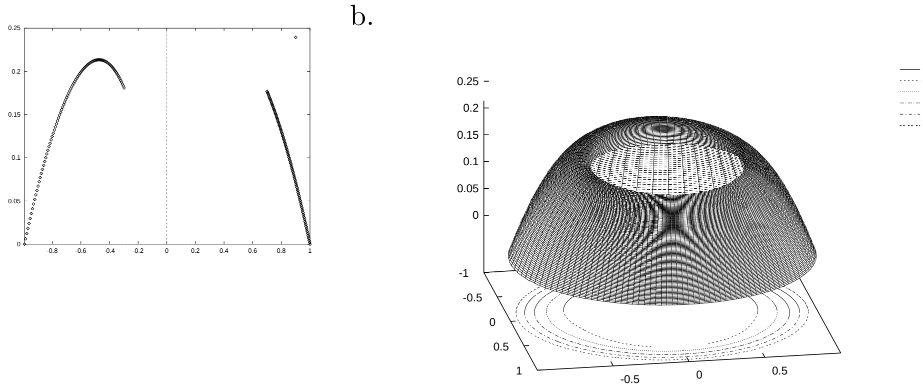

Figure 3: Case $j = 0$ with $c_0 = 0.2$ and $R_0 = 0.4$.

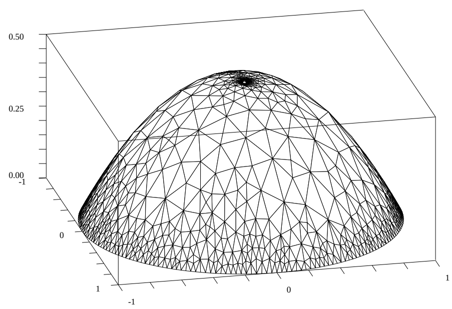

Figure 4: Case $j = 0$ with $c_0 = 0.1$ and $R_0 = 0.01$, by finite element technique.

a. b.

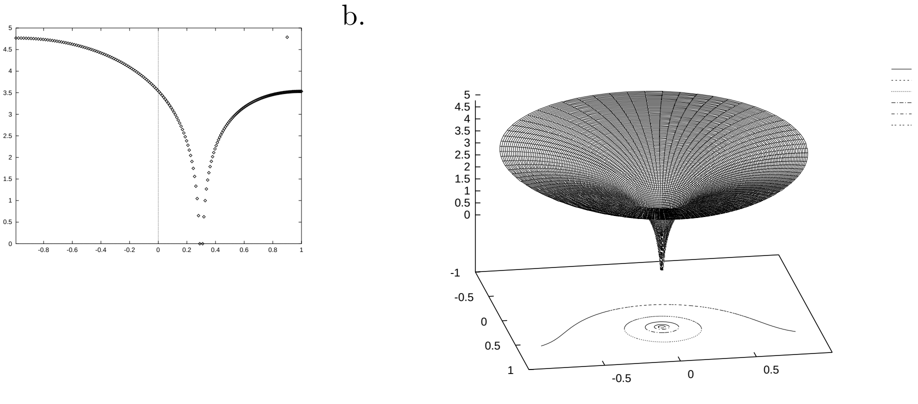

Figure 5: Case $j = 1$ with $c_0 = 0.3$ and $R_0 = 0.01$.

a. b.

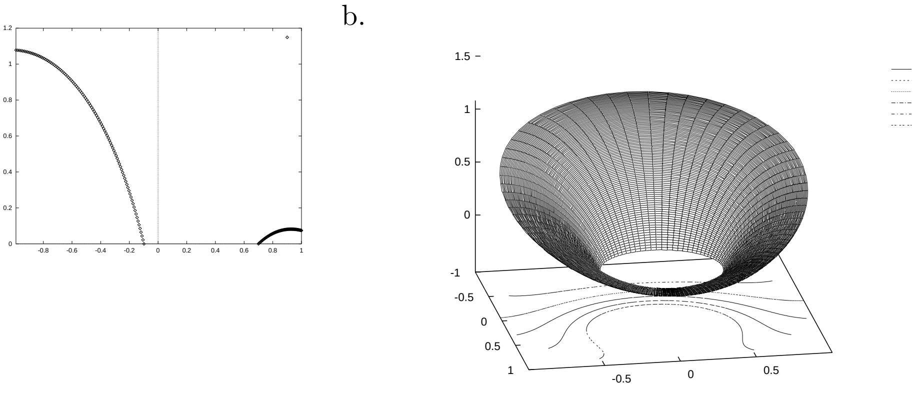

Figure 6: Case $j = 1$ with $c_0 = 0.3$ and $R_0 = 0.4$.

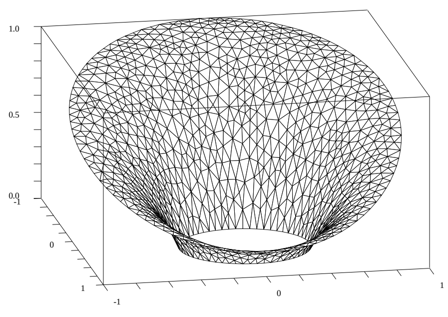

Figure 7: Case $j = 1$ with $c_0 = 0.3$ and $R_0 = 0.4$, by finite element technique.

APPENDIX

We give here the proofs of lemmas which we used.

Proof of Lemma 2.5. Let us denote $\zeta := \rho e^{i\theta}$, $\lambda := \sigma e^{i\tau}$ and for $\sigma \neq 0$, $a := (\rho/\sigma)\sin\tau$ and $b = (\rho/\sigma)\cos\tau$. We can write

$$I := \int_{A_{R_0}} \log|\zeta-\lambda|^2 d\zeta = \int_{R_0}^1 \rho d\rho \int_0^{2\pi} d\theta\, \log(\rho^2+\sigma^2-2\rho\sigma\cos\theta\cos\tau-2\rho\sigma\sin\theta\sin\tau)$$

$$= \int_{R_0}^1 \rho d\rho \left(2\pi\log\sigma^2 + \int_0^{2\pi} d\theta\, \log(1+a^2+b^2-2\,a\cos\theta-2\,b\sin\theta)\right).$$

Since,

$$\int_0^{2\pi} d\theta\, \log(1+a^2+b^2-2a\cos\theta-2b\sin\theta) = 2\pi \mathbb{1}_{\{a^2+b^2\geq 1\}} \log(a^2+b^2),$$

(see also [5] p. 528, 4.225.4), we obtain

$$I = 2\pi \int_{R_0}^1 \rho d\rho \left(\log\sigma^2 + \mathbb{1}_{\{\rho\geq\sigma\}} \log\left(\rho^2/\sigma^2\right)\right)$$

$$= \begin{cases} 2\pi(\int_{R_0}^1 \rho d\rho)\log\sigma^2 & \text{if } \sigma > 1 \\ 2\pi(\int_{R_0}^{\sigma} \rho d\rho)\log\sigma^2 + 2\pi\int_\sigma^1 \rho\log\rho^2 d\rho, & \text{if } R_0 < \sigma < 1 \\ 2\pi\int_{R_0}^1 \rho\log\rho^2 d\rho, & \text{if } \sigma < R_0. \end{cases}$$

The preceding equality lies also for $\sigma = 0$, and we get (2.5). □

Proof of Lemma 3.4. Let us denote $\zeta := \rho e^{i\theta}$ and $a := \rho/k < 1$. We can write,

$$\int_{\{|\zeta|=\rho\}} \frac{\partial}{\partial \mathrm{n}} \frac{1}{|\zeta-k|^2} d\sigma = \rho \int_0^{2\pi} d\theta\, \frac{\partial}{\partial\rho} \frac{1}{\rho^2+k^2-2k\rho\cos\theta}$$

$$= \frac{a}{k^2}\int_0^{2\pi} \frac{(-2a+2\cos\theta)d\theta}{(1+a^2-2a\cos\theta)^2} = \frac{a}{k^2}\frac{d}{da}\int_0^{2\pi} \frac{d\theta}{1+a^2-2a\cos\theta} = \frac{a}{k^2}\frac{d}{da}\frac{2\pi}{1-a^2}.$$

From this we get (3.4). □

For the proof of Lemma 3.5 we need the following result:

Lemma A.1. - *For* $\lambda \in \mathbb{C}$,

(A.1) $$\int_{\{|\zeta|=\rho\}} \frac{\log|\zeta-\lambda|^2}{|\zeta-k|^2} d\sigma = \frac{2\pi\rho}{k^2-\rho^2} \cdot \begin{cases} \log\left|\lambda-\rho^2/k\right|^2, & \text{if } |\lambda| > \rho \\ \log\rho^2\left|1-\lambda/k\right|^2, & \text{if } |\lambda| < \rho. \end{cases}$$

Proof. Let us denote $\zeta := \rho e^{i\theta}$, $\lambda := \sigma e^{i\tau}$, $a := \rho/k < 1$ and for $\sigma \neq 0$, $b := \rho/\sigma$. We can write

$$I := \int_{\{|\zeta|=\rho\}} \frac{\log|\zeta-\lambda|^2}{|\zeta-k|^2} d\sigma = \frac{a}{k}\int_0^{2\pi} d\theta\, \frac{\log\sigma^2}{1+a^2-2a\cos\theta}$$

$$+\frac{a}{k}\int_0^{2\pi} d\theta\, \frac{\log(1+b^2-2b\cos\theta\cos\tau-2b\sin\theta\sin\tau)}{1+a^2-2a\cos\theta}.$$

The first integral is equal to $(2\pi\log\sigma^2)/(1-a^2)$. Then (A.1) is easily obtained for real λ (that is for $\tau=0$ or $\tau=\pi$) since:

$$\int_0^{2\pi} d\theta\, \frac{\log(1+b^2-2b\cos\theta)}{1+a^2-2a\cos\theta} = \frac{4\pi}{1-a^2}\cdot\begin{cases} \log(1-ab), & \text{if } b^2\le 1 \\ \log(b-a), & \text{if } b^2>1, \end{cases}$$

(see also [5], p. 594, 4.397.16).

Then, by a classical argument of analytic continuation, we obtain

$$\int_{\{|\zeta|=\rho\}} \frac{\log(\zeta-\lambda)^2}{|\zeta-k|^2} d\sigma = \frac{2\pi\rho}{k^2-\rho^2}\cdot\begin{cases} \log\left(\lambda-\rho^2/k\right)^2, & \text{if } |\lambda|>\rho \\ \log\rho^2\left(1-\lambda/k\right)^2, & \text{if } |\lambda|<\rho. \end{cases}$$

The preceding equality lies also for $\sigma=|\lambda|=0$ and its real part is nothing but (A.1). $\square$

Proof of Lemma 3.5. We shall use the same notations as in the proof of Lemma A.1. We write $J(\rho,\lambda)=I_1-I_2$, where

$$I_1 := \int_{\{|\zeta|=\rho\}} \log|\zeta-\lambda|^2 \frac{\partial}{\partial \mathrm{n}}\frac{1}{|\zeta-k|^2} d\sigma$$

$$= \rho\int_0^{2\pi} d\theta\, \log(\rho^2+\sigma^2-2\rho\sigma\cos\theta\cos\tau-2\rho\sigma\sin\theta\sin\tau)\, \frac{\partial}{\partial\rho}\frac{1}{\rho^2+k^2-2k\rho\cos\theta}$$

$$= \frac{a}{k^2}\int_0^{2\pi} d\theta \log(\rho^2+\sigma^2-2\rho\sigma\cos\theta\cos\tau-2\rho\sigma\sin\theta\sin\tau)\, \frac{-2a+2\cos\theta}{(1+a^2-2a\cos\theta)^2}$$

$$= \frac{a}{k^2}\frac{\partial}{\partial a}\int_0^{2\pi} d\theta\, \frac{\log(\rho^2+\sigma^2-2\rho\sigma\cos\theta\cos\tau-2\rho\sigma\sin\theta\sin\tau)}{1+a^2-2a\cos\theta},$$

and

$$I_2 := \int_{\{|\zeta|=\rho\}} \frac{1}{|\zeta-k|^2}\frac{\partial}{\partial \mathrm{n}} \log|\zeta-\lambda|^2 d\sigma$$

$$= \rho\int_0^{2\pi} d\theta\, \frac{1}{\rho^2+k^2-2k\rho\cos\theta}\frac{\partial}{\partial\rho}\log(\rho^2+\sigma^2-2\rho\sigma\cos\theta\cos\tau-2\rho\sigma\sin\theta\sin\tau)$$

$$= b\int_0^{2\pi} d\theta\, \frac{1}{\rho^2+k^2-2k\rho\cos\theta}\cdot\frac{2b-2\cos\theta\cos\tau-2\sin\theta\sin\tau}{1+b^2-2b\cos\theta\cos\tau-2b\sin\theta\sin\tau}$$

$$= b\frac{\partial}{\partial b}\int_0^{2\pi} d\theta\, \frac{\log(1+b^2-2\,b\cos\theta\cos\tau-2\,b\sin\theta\sin\tau)}{\rho^2+k^2-2k\rho\cos\theta}.$$

But (A.1) can be written as

$$\text{(A.2)}\qquad \int_0^{2\pi} d\theta\, \frac{\log(\rho^2+\sigma^2-2\rho\sigma\cos\theta\cos\tau-2\rho\sigma\sin\theta\sin\tau)}{1+a^2-2a\cos\theta}$$

$$= \frac{2\pi}{1-a^2}\cdot\begin{cases} \log\rho^2(1/b^2+a^2-2(a/b)\cos\tau), & \text{if } |\lambda|>\rho \\ \log\rho^2(1+a^2/b^2-2(a/b)\cos\tau), & \text{if } |\lambda|<\rho, \end{cases}$$

or as

$$\text{(A.3)} \qquad \int_0^{2\pi} d\theta \, \frac{\log(1+b^2-2b\cos\theta\cos\tau-2b\sin\theta\sin\tau)}{\rho^2+k^2-2k\rho\cos\theta}$$

$$= \frac{2\pi}{k^2-\rho^2}\cdot\begin{cases} \log(1+a^2b^2-2ab\cos\tau), & \text{if } |\lambda|>\rho \\ \log(b^2+a^2-2ab\cos\tau), & \text{if } |\lambda|<\rho. \end{cases}$$

By derivation of (A.2) and (A.3) with respect to a and b respectively, we get

$$I_1 = \begin{cases} \frac{4\pi\rho^2}{k^2(k^2-\rho^2)}\cdot\frac{\rho^2-|\lambda|k\cos\tau}{|\lambda-\rho^2/k|^2}+\frac{8\pi\rho^2}{(k^2-\rho^2)^2}\log\left|\lambda-\rho^2/k\right|, & \text{if } |\lambda|>\rho \\ \\ \frac{4\pi}{k^2(k^2-\rho^2)}\cdot\frac{|\lambda|(|\lambda|-k\cos\tau)}{|1-\lambda/k|^2}+\frac{8\pi\rho^2}{(k^2-\rho^2)^2}\log\rho\,|1-\lambda/k|, & \text{if } |\lambda|<\rho, \end{cases}$$

and

$$I_2 = \begin{cases} \frac{4\pi\rho^2}{k^2(k^2-\rho^2)}\cdot\frac{\rho^2-|\lambda|k\cos\tau}{|\lambda-\rho^2/k|^2}, & \text{if } |\lambda|>\rho \\ \\ \frac{4\pi}{(k^2-\rho^2)}\cdot\frac{1-|\lambda/k|\cos\tau}{|1-\lambda/k|^2}, & \text{if } |\lambda|<\rho \end{cases}$$

and the proof of Lemma 3.5 is done. □

Acknowledgements

The authors are grateful to Bernard Roynette for stimulating discussions and helpful comments, and to Jean-Philippe Anker and Pierre Vallois for theirs suggestions.

References

[1] L. Ahlfors, *Complex Analysis*, New York: McGraw-Hill Book Company, 1966

[2] R.B. Burckel, *An Introduction to Classical Complex Analysis, I*, New York, San Francisco: Academic Press, 1979

[3] R. Courant and D. Hilbert, *Methods of Mathematical Physics, I*, New York: Interscience Publishers Inc., 1953

[4] E.B. Dynkin and A.A. Yushkevich, *Markov Processes, Theorems and Problems* New York: Plenum Press, 1969

[5] I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, Series and Products*, New York: Academic Press, 1980

[6] P. Henrici, *Applied and Computational Complex Analysis, III*, New York London, Sydney, Toronto: Wiley Interscience, 1986

[7] E. Hille, *Analytic Function Theory, I*, Boston: Ginn and Company, 1959

[8] K. Ito and H.P. McKean Jr., *Diffusion Processes and their Sample Paths*, Berlin, Heidelberg, New York: Springer Verlag, 1974

[9] I. Karatzas and S.E. Shreve, *Brownian Motion and Stochastic Calculus*, Berlin, Heidelberg, New York: Springer-Verlag, 1991

[10] P.L. Lions and A.S. Sznitman, *Stochastic Differential Equations with Reflecting Boundary Conditions*, Vol. **37**, 1984, pp. 511-537

[11] M. Rao, *Brownian Motion and Classical Potential Theory*, Matematik Institut of Aarhus Universitet: Lecture Notes Series, No. 47, 1977

[12] J.R. Roche and J. Sokolowski, Numerical methods for shape identification problems, *Control and Cybernetics*, Vol. **25**, 1996, pp. 867-894

[13] H. Villat, Le problème de Dirichlet dans une aire annulaire, *Rendic. Circ. Mat. Palermo*, Vol. **33**, 1912, pp. 134-175.